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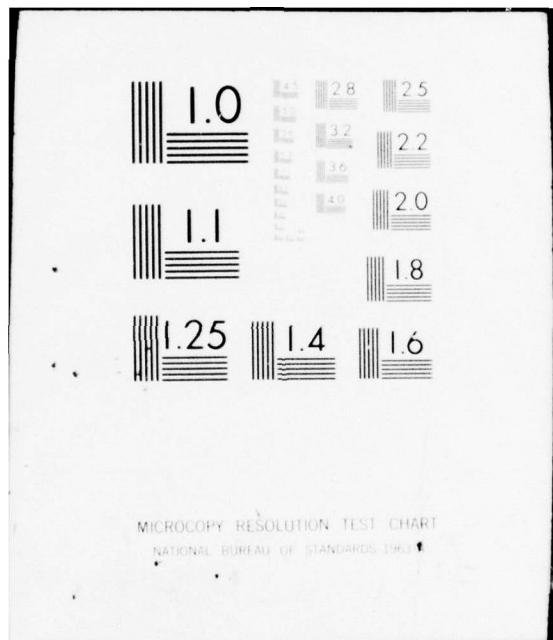
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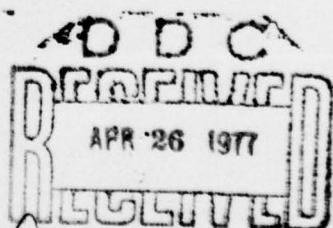
NWC TP 5900

Transmission of Monochromatic Electromagnetic Waves Through Systems Consisting of Multiple Layers of Materials Exhibiting Birefringence or Faraday Rotation

by
David J. White
Research Department

MARCH 1977

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R. G. Freeman, III, RAdm., USN Commander

G. L. Hollingsworth Technical Director

FOREWORD

The work contained in this report represents part of a continuing effort in millimeter and microwave research by the Physical Electronics Branch of the Physics Division, Naval Weapons Center. The treatment of wave propagation described is also applicable to infrared and optical wave propagation because of the unity of electromagnetic wave propagation regardless of wavelength.

The work was funded under Independent Research Funding and as such represents part of the effort to support NWC (and Navy) systems and development work with a thorough understanding of physical principles and their manipulation in order to design and fabricate new and better components and systems.

This report has been reviewed for technical accuracy by M. Hibbard and M. L. Scott.

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(U) The problem of plane electromagnetic wave propagation through systems consisting of plates or layers of birefringent or gyroelectric materials is considered. Scattering matrices for the two types of layers are given, where propagation is normal to the plates and along a natural axis or the Faraday direction. The equations for converting the 4×4 scattering matrices to 4×4 transmission matrices are given. The overall transmission matrix for a system is then found by multiplying the appropriate coordinate transformation matrices together with the transformation matrices of the individual layers. Various optical and microwave examples of isolators and phase shifters are worked out.



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I. INTRODUCTION

This paper is written with a two-fold purpose. First, with the passage of time, applications (filters, nonreciprocal devices, modulators, switches, etc.) involving layered structures containing both birefringent and gyrotropic materials appear likely to become more common (Ref. 1-7). Indeed, the Faraday rotation isolator with polarizer, 45-degree Faraday rotating section, 45-degree polarizer is such a system and is in widespread use--at least in its microwave analog of rectangular waveguide, cylindrical waveguide containing ferrite, and 45-degree rectangular waveguide.

Here we would like to present a method of dealing with a system consisting of an arbitrary number of mixed birefringent and gyrotropic layers including all multiple reflections. The conditions are:

1. Coherent electromagnetic plane waves at normal incidence.
2. Propagation to be along the direction of a principal axis in each of the birefringent layers, and along the Faraday axis (i.e., in the direction of the applied magnetic field) in the gyrotropic layers.
3. The gyrotropic layers have no intrinsic anisotropy (birefringence) of their own.
4. The permeability is assumed isotropic.

To our knowledge, no simple rigorous treatment of such a system has appeared (although the "Scientific Literature Explosion" of recent years should be kept in mind); however, References 8-25 appear generally pertinent.*

The second reason for this paper lies in what seems to have been a source of confusion among at least some scientists--including the author. Most of us have been exposed to the idea that conservation of energy requires that matrices representing dielectric constant, conductivity, and permeability be symmetric--i.e., $\epsilon_{ij} = \epsilon_{ji}$. (For example, see Ref. 26). Furthermore, we are shown, the scattering matrix is symmetrical--assuming the matrix elements are properly normalized to the impedance levels on both sides of the obstacle (Ref. 27).

* References 23-25 are available from the author on request.

Figure 1 shows the reflection and transmission of electromagnetic waves from a general obstacle. However, if we write as a black-box approach to Figure 1:

$$\begin{bmatrix} {}^L E_1 \\ {}^R E_2 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} {}^R E_1 \\ {}^L E_2 \end{bmatrix} \quad (1)$$

and apply it to a microwave isolator as a two-port device, the scattering matrix (for a perfect isolator) appears to be

$$\begin{bmatrix} 0 & 0 \\ e^{j\phi} & 0 \end{bmatrix} \quad (2)$$

It is somewhat difficult to call this matrix symmetrical.*

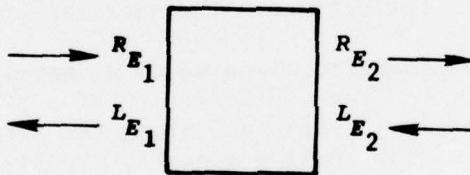


FIGURE 1. Reflection and Transmission of Electromagnetic Waves From a General Obstacle.

It is part of the purpose of this report, then, to show what happens to the symmetry of the scattering matrices when written in coordinate systems proper for the normal modes of propagation in the particular materials involved.

* The notation super R , L, E , Sub 1,2 is used rather than the more conventional E , Sub 1,2, and E' , Sub 1,2, representing incident and reflected (prime) waves on the left (1) side or right (2) side of a barrier. Here R refers to right propagating and L to left propagating.

II. DERIVATIONS

PART A: THE BIREFRINGENT PLATE

Consider Figure 2. Plane waves propagating in the $\pm z$ directions are normally incident on a semi-infinite slab of birefringent material whose principal axes lie along the x , y , z axes characterized by the dielectric constants ϵ_x , ϵ_y , ϵ_z . The dielectric constant matrix is thus: (Ref. 26)

$$\begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \quad (3)$$

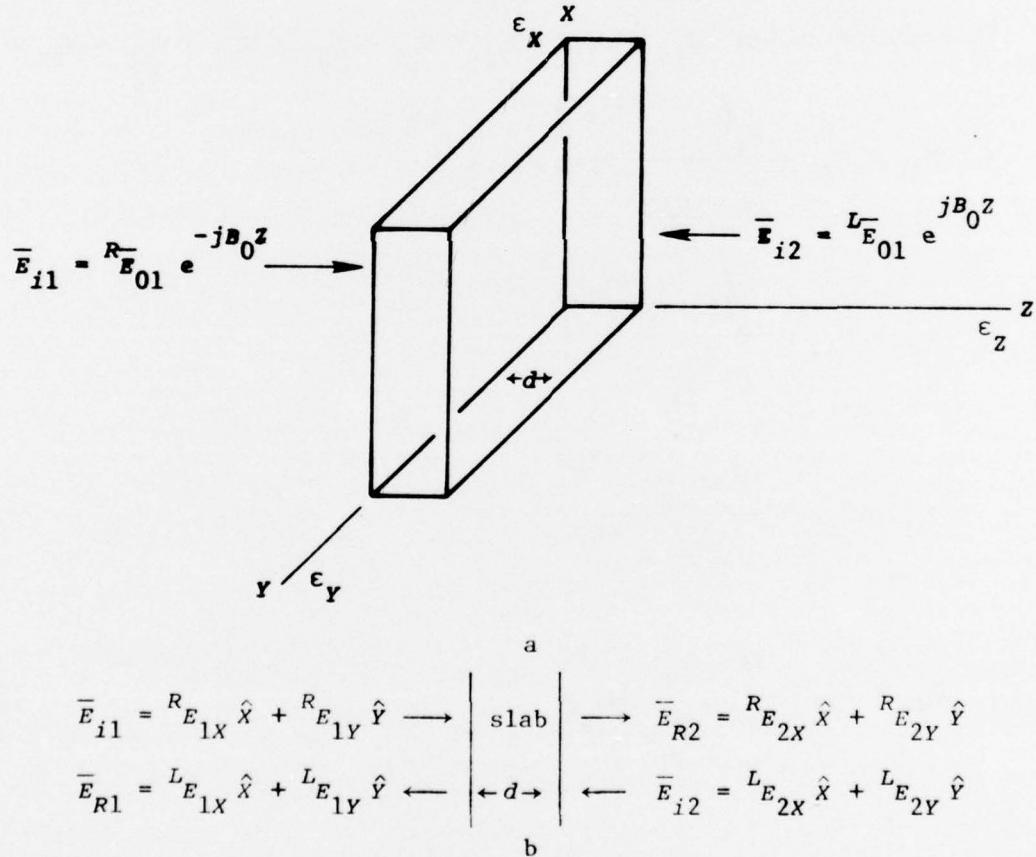


FIGURE 2. (a) Plane Waves at Normal Incidence on a Birefringent Slab Such That the Direction of Propagation Is Along a Principal Axis; (b) Schematic Representation With Input Waves Broken Up Into Incident and Reflected Components.

Waves polarized along the X and Y axes propagate in the slab with different velocities and attenuation, but unchanged in form. These are the normal modes in propagation along a principal axis in birefringent material as may be verified (Ref. 23) by reference to Maxwell's equations.

In scattering matrix notation, this situation can be represented by:

$$\begin{bmatrix} L_E_{1X} \\ R_E_{2X} \\ L_E_{1Y} \\ R_E_{2Y} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} R_E_{1X} \\ L_E_{2X} \\ R_E_{1Y} \\ L_E_{2Y} \end{bmatrix} \quad (4)$$

Examination shows: $S_{13} = S_{14} = S_{23} = S_{24} = S_{31} = S_{32} = S_{41} = S_{42} = 0$
and:

$$\begin{aligned} S_{11} = S_{22} &= \frac{r_x \left(1 - e^{-2\gamma_x d} \right)}{1 - r_x^2 e^{-2\gamma_x d}} \\ S_{33} = S_{44} &= \frac{r_y \left(1 - e^{-2\gamma_y d} \right)}{1 - r_y^2 e^{-2\gamma_y d}} \\ S_{12} = S_{21} &= \frac{(1 - r_x^2) e^{-\gamma_x d}}{1 - r_x^2 e^{-2\gamma_x d}} \\ S_{34} = S_{43} &= \frac{(1 - r_y^2) e^{-\gamma_y d}}{1 - r_y^2 e^{-2\gamma_y d}} \end{aligned} \quad (5)$$

where

$$r_{x,y} = \frac{\eta_{x,y} - 1}{\eta_{x,y} + 1}$$

$$\eta_{x,y} = \sqrt{\frac{\mu}{\epsilon_{x,y}}}$$

$$\gamma_{x,y} = j \frac{\omega}{c} \sqrt{\mu \epsilon_{x,y}}$$

Equation (4) thus becomes:

$$\begin{bmatrix} L_{E_{1x}} \\ R_{E_{2x}} \\ L_{E_{1y}} \\ R_{E_{2y}} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{12} & s_{11} & 0 & 0 \\ 0 & 0 & s_{33} & s_{34} \\ 0 & 0 & s_{34} & s_{33} \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{2x}} \\ R_{E_{1y}} \\ L_{E_{2y}} \end{bmatrix} \quad (6)$$

Other arrangements of the scattering elements are possible depending on the arrangement of the electric field components in the column vectors, but this one appears appropriate in that it is symmetrical and clearly shows the separate nature of the normal modes.

In dealing with a series of slabs, scattering matrices are inadequate in that they cannot be multiplied together by the rules of standard matrix multiplication to obtain an overall scattering matrix--although a special definition (Ref. 28) of matrix multiplication is possible.

For our purposes we rewrite Equation (4) as:

$$\begin{bmatrix} R_{E_{2x}} \\ L_{E_{2x}} \\ R_{E_{2y}} \\ L_{E_{2y}} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (7)$$

where:

$$t_{21} = \frac{s_{14}s_{31}-s_{11}s_{34}}{s_{12}s_{34}-s_{14}s_{32}}$$

$$t_{22} = \frac{s_{34}}{s_{12}s_{34}-s_{14}s_{32}}$$

$$t_{23} = \frac{s_{14}s_{33}-s_{13}s_{34}}{s_{12}s_{34}-s_{14}s_{32}}$$

$$t_{24} = \frac{-s_{14}}{s_{12}s_{34}-s_{14}s_{32}}$$

$$t_{41} = \frac{s_{12}s_{31}-s_{11}s_{32}}{s_{14}s_{32}-s_{12}s_{34}}$$

$$t_{42} = \frac{s_{32}}{s_{14}s_{32}-s_{12}s_{34}}$$

$$t_{43} = \frac{s_{12}s_{33}-s_{13}s_{32}}{s_{14}s_{32}-s_{12}s_{34}}$$

$$t_{44} = \frac{-s_{12}}{s_{14}s_{32}-s_{12}s_{34}}$$

$$t_{11} = s_{21} + s_{22}t_{21} + s_{24}t_{41}$$

$$t_{12} = s_{22}t_{22} + s_{24}t_{42}$$

$$t_{13} = s_{23} + s_{22}t_{23} + s_{24}t_{43}$$

$$t_{14} = s_{22}t_{24} + s_{24}t_{44}$$

$$t_{31} = s_{41} + s_{42}t_{21} + s_{44}t_{41}$$

$$t_{32} = s_{42}t_{22} + s_{44}t_{42}$$

$$t_{33} = s_{43} + s_{42}t_{23} + s_{44}t_{43}$$

$$t_{34} = s_{42}t_{24} + s_{44}t_{44}$$

(8)

Going from the transmission (T) matrix to the (S) matrix:

$$s_{11} = \frac{t_{24}t_{41} - t_{21}t_{44}}{t_{22}t_{44} - t_{24}t_{42}}$$

$$s_{12} = \frac{t_{44}}{t_{22}t_{44} - t_{24}t_{42}}$$

$$s_{13} = \frac{t_{24}t_{43} - t_{23}t_{44}}{t_{22}t_{44} - t_{24}t_{42}}$$

$$s_{14} = \frac{-t_{24}}{t_{22}t_{44} - t_{24}t_{42}}$$

$$s_{31} = \frac{t_{22}t_{41} - t_{21}t_{42}}{t_{24}t_{42} - t_{22}t_{44}}$$

$$s_{32} = \frac{t_{42}}{t_{24}t_{42} - t_{22}t_{44}}$$

$$s_{33} = \frac{t_{22}t_{43} - t_{23}t_{42}}{t_{24}t_{42} - t_{22}t_{44}}$$

$$s_{34} = \frac{-t_{22}}{t_{24}t_{42} - t_{22}t_{44}}$$

$$s_{21} = t_{11} + t_{12}s_{11} + t_{14}s_{31}$$

$$s_{22} = t_{12}s_{12} + t_{14}s_{32}$$

$$s_{23} = t_{13} + t_{12}s_{13} + t_{14}s_{33}$$

$$s_{24} = t_{12}s_{14} + t_{14}s_{34}$$

$$s_{41} = t_{31} + t_{32}s_{11} + t_{34}s_{31}$$

$$s_{42} = t_{32}s_{12} + t_{34}s_{32}$$

$$s_{43} = t_{33} + t_{32}s_{13} + t_{34}s_{33}$$

$$s_{44} = t_{32}s_{14} + t_{34}s_{34}$$

(9)

For the particular case of the birefringent plate (Equation 6), Equation 8 reduces to:

$$\begin{aligned}
 t_{21} &= -S_{11}/S_{12} = \frac{-2r_x}{1 - r_x^2} e^{-\gamma_x d} \sinh \gamma_x d \\
 t_{22} &= 1/S_{12} = \frac{1 - r_x^2 e^{-2\gamma_x d}}{(1 - r_x^2) e^{-\gamma_x d}} \\
 t_{23} &= t_{24} = t_{41} = t_{42} = 0 \\
 t_{43} &= -S_{33}/S_{34} = \frac{-2r_y}{1 - r_y^2} e^{-\gamma_y d} \sinh \gamma_y d \\
 t_{44} &= 1/S_{34} = \frac{1 - r_y^2 e^{-2\gamma_y d}}{(1 - r_y^2) e^{-\gamma_y d}} \\
 t_{11} &= \frac{S_{12}^2 - S_{11}^2}{S_{12}} = \frac{e^{-\gamma_x d} - r_x^2 e^{\gamma_x d}}{1 - r_x^2} \\
 t_{12} &= S_{11}/S_{12} = -t_{21} \\
 t_{13} &= t_{14} = t_{31} = t_{32} = 0 \\
 t_{33} &= \frac{S_{34}^2 - S_{33}^2}{S_{34}} = \frac{e^{-\gamma_y d} - r_y^2 e^{\gamma_y d}}{1 - r_y^2} \\
 t_{34} &= S_{33}/S_{34} = -t_{43}
 \end{aligned} \tag{10}$$

Rewriting Equation 7 for this case:

$$\begin{bmatrix} R_{E_{2x}} \\ L_{E_{2x}} \\ R_{E_{2y}} \\ L_{E_{2y}} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & 0 & 0 \\ -t_{12} & t_{22} & 0 & 0 \\ 0 & 0 & t_{33} & t_{34} \\ 0 & 0 & -t_{34} & t_{44} \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \tag{11}$$

PART B: THE FARADAY PLATE

Consider the case of Faraday rotation. The normal modes for propagation along the Faraday axis are of the form (Ref. 18, 19, 24, 25):

$$\begin{aligned}\bar{E}_+ &= E_+ (\hat{x} + j\hat{y}) e^{\pm \gamma_+ z} \\ \bar{E}_- &= E_- (\hat{x} - j\hat{y}) e^{\pm \gamma_- z}\end{aligned}\quad (12)$$

The modes $\hat{x} + j\hat{y}$ and $\hat{x} - j\hat{y}$ are completely decoupled (Ref. 24, 25) and preserve their form regardless of the direction of propagation--thus saving any worry over the concept of left- and right-handed circular polarization and "Who may be looking in what direction."

In scattering matrix notation for Figure 3:

$$\begin{bmatrix} L_{E_{1+}} \\ R_{E_{2+}} \\ L_{E_{1-}} \\ R_{E_{2-}} \end{bmatrix} = (S_f) \begin{bmatrix} R_{E_{1+}} \\ L_{E_{2+}} \\ R_{E_{1-}} \\ L_{E_{2-}} \end{bmatrix} \quad (13)$$

The coefficients for the S matrix are (Ref. 24, 25):

$$S_{13} = S_{14} = S_{23} = S_{24} = S_{31} = S_{32} = S_{41} = S_{42} = 0$$

$$S_{11} = S_{22} = \frac{r_+ (1 - e^{-2\gamma_+ d})}{1 - r_+^2 e^{-2\gamma_+ d}}$$

$$S_{33} = S_{44} = \frac{r_- (1 - e^{-2\gamma_- d})}{1 - r_-^2 e^{-2\gamma_- d}}$$

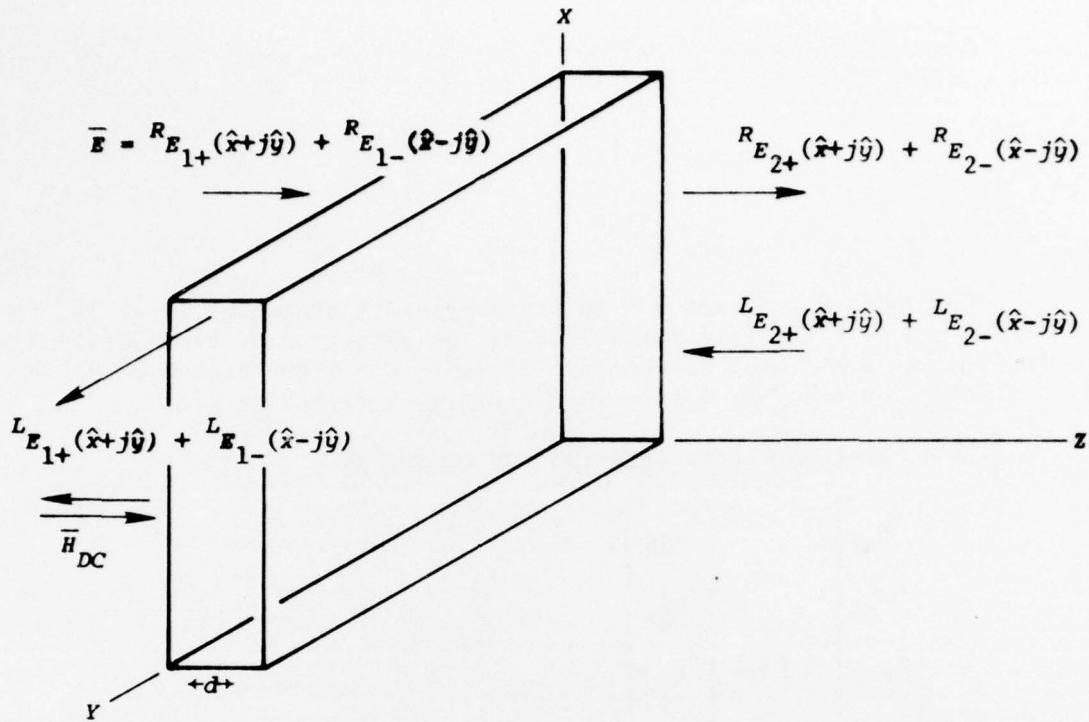


FIGURE 3. Plane Waves at Normal Incidence on a Faraday Plate Such That the Direction of Propagation Is Along the Faraday Axis.

$$\begin{aligned}
 S_{12} = S_{21} &= \frac{(1 - r_+^2) e^{-\gamma_+ d}}{1 - r_+^2 e^{-2\gamma_+ d}} \\
 S_{34} = S_{43} &= \frac{(1 - r_-^2) e^{-\gamma_- d}}{1 - r_-^2 e^{-2\gamma_- d}}
 \end{aligned} \tag{14}$$

where

$$r_{\pm} = \frac{n_{\pm} - 1}{n_{\pm} + 1}$$

$$\gamma_{\pm} = j \frac{\omega}{c} \sqrt{\mu\epsilon_{\pm}}$$

For unity permeability, " μ " and a dielectric constant matrix of the form

$$(\epsilon_c) = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ -\epsilon_{xy} & \epsilon_{xx} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \quad (15)$$

Then

$$\begin{aligned} r_{\pm} &= \frac{\gamma_o - \gamma_{\pm}}{\gamma_o + \gamma_{\pm}} \\ \gamma_{\pm}^2 &= \omega^2 \mu \epsilon_o (\epsilon_{xx} \mp j \epsilon_{xy}) \end{aligned} \quad (16)$$

Equation 13 becomes

$$\begin{bmatrix} L_{E_{1+}} \\ R_{E_{2+}} \\ L_{E_{1-}} \\ R_{E_{2-}} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & 0 & 0 \\ s_{12} & s_{11} & 0 & 0 \\ 0 & 0 & s_{33} & s_{34} \\ 0 & 0 & s_{34} & s_{33} \end{bmatrix} \begin{bmatrix} R_{E_{1+}} \\ L_{E_{2+}} \\ R_{E_{1-}} \\ L_{E_{2-}} \end{bmatrix} \quad (17)$$

Equation 13 can be rewritten in the transmission matrix format:

$$\begin{bmatrix} R_{E_{2+}} \\ L_{E_{2+}} \\ R_{E_{2-}} \\ L_{E_{2-}} \end{bmatrix} = (T_f) \begin{bmatrix} R_{E_{1+}} \\ L_{E_{1+}} \\ R_{E_{1-}} \\ L_{E_{1-}} \end{bmatrix} \quad (18)$$

Since Equation 13 and 4 as well as 18 and 7 are in the same form with \pm substituted for x, y , the coefficients t_{ij} are given by Equation 8 and s_{ij} by Equation 9.

In going from a birefringent plate where the normal modes are orthogonal linearly polarized waves to a Faraday plate where the normal modes are \pm circularly polarized, some sort of transformation is needed.

Consider a general elliptically polarized wave:

$$\bar{E} = E_x \hat{x} + E_y \hat{y} \quad (19)$$

where E_x and E_y are in general complex, $(E_{x0} e^{\pm \gamma_x z}, E_{y0} e^{\pm \gamma_y z})$.

We wish to transform to:

$$\bar{E} = E_+ (\hat{x} + j\hat{y}) + E_- (\hat{x} - j\hat{y}) = (E_+ + E_-) \hat{x} + j(E_+ - E_-) \hat{y} \quad (20)$$

(Again $E_{\pm} = E_{0\pm} e^{\frac{\pm \gamma_z}{2} z}$.)

Setting:

$$\begin{aligned} E_x &= E_+ + E_- \\ E_y &= j(E_+ - E_-) \end{aligned} \quad (21)$$

Yields:

$$\begin{aligned} E_+ &= \frac{1}{2}(E_x - jE_y) \\ E_- &= \frac{1}{2}(E_x + jE_y) \end{aligned} \quad (22)$$

Or in matrix form:

$$\begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \quad (23)$$

The inverse of this transformation is readily shown to be:

$$\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \quad (24)$$

This brings up a point mentioned in the introduction. Consider a statement of Ohm's law for a material exhibiting Faraday rotation or Hall effect due to an antisymmetrical conductivity matrix:

$$\begin{bmatrix} J_x \\ J_y \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \quad (25)$$

Clearly:

$$\begin{bmatrix} J_+ \\ J_- \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -j \\ 1 & j \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} E_+ \\ E_- \end{bmatrix} \quad (26)$$

Taking $\frac{1}{2}$ times the first three matrices in the right-hand side of the last equation results in:

$$\begin{bmatrix} \sigma_+ & 0 \\ 0 & \sigma_- \end{bmatrix} = \begin{bmatrix} \sigma_{xx} + j\sigma_{xy} & 0 \\ 0 & \sigma_{xx} - j\sigma_{xy} \end{bmatrix} \quad (27)$$

which is symmetrical, verifying that the conductivity matrix is symmetrical in the proper (\pm) coordinate system.

In terms of Figure 4, a birefringent plate followed by a Faraday Rotater, the following sequence can now be built:

$$\begin{bmatrix} R_{E_{2x}} \\ L_{E_{2x}} \\ R_{E_{2y}} \\ L_{E_{2y}} \end{bmatrix} = (\mathcal{T}_B) \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (28)$$

where (\mathcal{T}_B) is given by Equation 10 and 11. However,

$$\begin{bmatrix} R_{E_{2+}} \\ L_{E_{2+}} \\ R_{E_{2-}} \\ L_{E_{2-}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & -j & 0 \\ 0 & 1 & 0 & -j \\ 1 & 0 & j & 0 \\ 0 & 1 & 0 & j \end{bmatrix} \begin{bmatrix} R_{E_{2x}} \\ L_{E_{2x}} \\ R_{E_{2y}} \\ L_{E_{2y}} \end{bmatrix} = (\mathcal{T}_f)(\mathcal{T}_B) \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (29)$$

where \mathcal{T}_f is the transformation matrix from x,y to \pm coordinates.

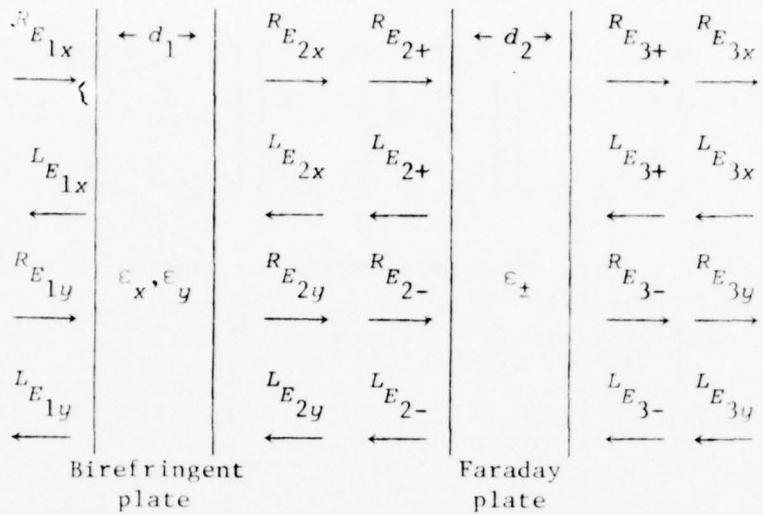


FIGURE 4. A Birefringent Medium Followed by a Faraday Medium, Incident and Reflected Plane Waves.

It then follows that:

$$\begin{bmatrix} R_{E_{3+}} \\ L_{E_{3+}} \\ R_{E_{3-}} \\ L_{E_{3-}} \end{bmatrix} = (T_f) (T_f) (T_B) \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (30)$$

Since

$$\begin{bmatrix} R_{E_{3x}} \\ L_{E_{3x}} \\ R_{E_{3y}} \\ L_{E_{3y}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ j & 0 & -j & 0 \\ 0 & j & 0 & -j \end{bmatrix} \begin{bmatrix} R_{E_{3+}} \\ L_{E_{3+}} \\ R_{E_{3-}} \\ L_{E_{3-}} \end{bmatrix} = (T_f)^{-1} \begin{bmatrix} R_{E_{3+}} \\ L_{E_{3+}} \\ R_{E_{3-}} \\ L_{E_{3-}} \end{bmatrix} \quad (31)$$

it follows that

$$\begin{bmatrix} R_{E_{3x}} \\ L_{E_{3x}} \\ R_{E_{3y}} \\ L_{E_{3y}} \end{bmatrix} = (T_f)^{-1} (T_f) (T_f) (T_B) \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (32)$$

It has been assumed that the plates were adjacent without intervening dielectric. The presence of an air or dielectric spacer would not pose any particular difficulty since an isotropic dielectric can be considered as either a degenerate Faraday plate where $\gamma_+ = \gamma_-$ or birefringent medium when $\gamma_x = \gamma_y$. Thus, for this case the scattering matrix reduces from Equation 17 or 6 to a case where $S_{11} = S_{33}$ and $S_{12} = S_{34}$. The transmission matrix for a dielectric spacer is found from Equation 11 with $t_{34} = t_{12}$, $t_{44} = t_{22}$, and $t_{11} = t_{33}$. This matrix must then be inserted between (T_f) and (T_B) in Equation 32.*

If, instead of a dielectric layer, there is space between the two plates, the reflection coefficients S_{11} and S_{33} go to zero and

$S_{12} = S_{34} = \epsilon^{-\gamma_0 d}$. This is because, in effect, all of our reflection and transmission coefficients have been normalized to free space. The transmission matrix is then (from Equation 10 and 11) $t_{12} = t_{34} = 0$, $t_{44} = t_{22} = \epsilon^{-\gamma_0 d}$, $t_{11} = t_{33} = \epsilon^{-\gamma_0 d}$.

* Since the isotropic dielectric is a limiting case for either a Faraday or birefringent medium, it could equally well be inserted between (T_f) and (T_f) in Equation 32. This is readily verified by observing:

$$(T_f) \begin{bmatrix} t_{11} & t_{12} & 0 & 0 \\ -t_{12} & t_{22} & 0 & 0 \\ 0 & 0 & t_{11} & t_{12} \\ 0 & 0 & -t_{12} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & 0 & 0 \\ -t_{12} & t_{22} & 0 & 0 \\ 0 & 0 & t_{11} & t_{12} \\ 0 & 0 & t_{12} & t_{22} \end{bmatrix} (T_f) = \begin{bmatrix} t_{11} & t_{12} & -jt_{11} & -jt_{12} \\ -t_{12} & t_{22} & jt_{12} & -jt_{22} \\ t_{11} & t_{12} & jt_{11} & jt_{12} \\ -t_{12} & t_{22} & -jt_{12} & jt_{22} \end{bmatrix}$$

As more plates are added, the multiplication of a string of 4×4 matrices becomes quickly out of hand; however, modern computers handle such matrix multiplications easily enough.

PART C: ROTATION OF COORDINATES FOR BIREFRINGENT PLATES

In the introduction it was specified only that propagation be along a principal axis of a birefringent plate. It is likely, however, that in a system of interest that various slabs will have their other two axes rotated with respect to each other--as in Figure 5.

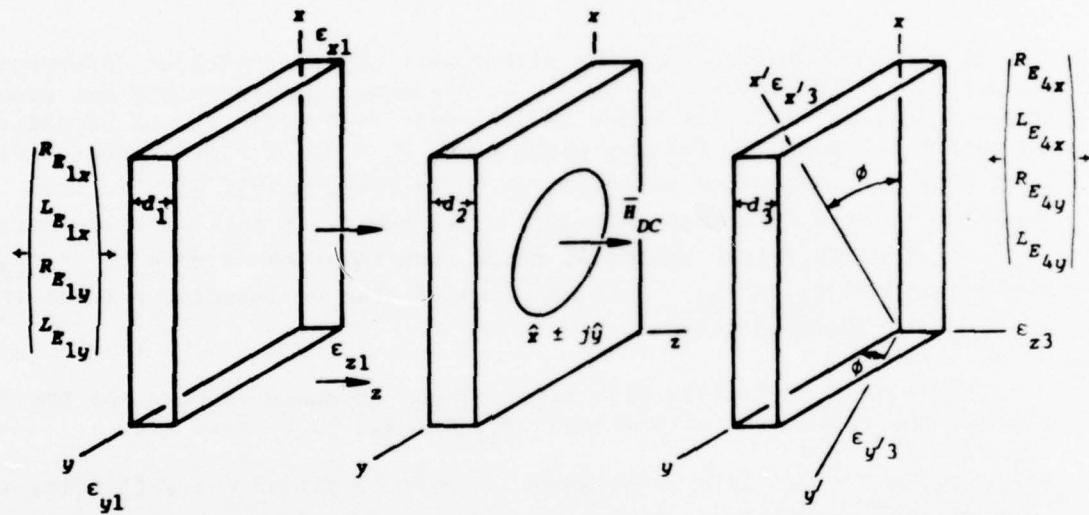


FIGURE 5. Birefringent Plate Followed by a Faraday Plate Followed by Another Birefringent Plate With Its Principal Axes Rotated at an Angle ϕ to the Original x,y Axes.

If the permittivity matrix for the third plate is written in terms of the principal x,y axes of the first plate there will result the form:

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ \epsilon_{xy} & \epsilon_{yy} & 0 \\ 0 & 0 & \epsilon_{zz} \end{bmatrix} \quad (33)$$

The solutions to the wave equation will no longer be in terms of E_x and E_y as normal modes; rather the normal modes will be counter-rotating elliptically polarized waves (Ref. 19, 23).

It is quite possible to write a transformation matrix (and its inverse) from x, y coordinates to counter-rotating elliptical coordinates, similar to the procedure followed for a Faraday plate. It seems conceptually simpler, however, to transform the x, y coordinates to the x', y' coordinates so that the scattering and transmission matrices (Equation 6 and 11) may be used directly with x and y replaced by x' and y' .

The transformation from xy to $x'y'$ is

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (34)$$

with an inverse

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Thus, we have

$$\begin{bmatrix} R_{E_{x'}} \\ L_{E_{x'}} \\ R_{E_{y'}} \\ L_{E_{y'}} \end{bmatrix} = \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & \cos \phi & 0 & \sin \phi \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & -\sin \phi & 0 & \cos \phi \end{bmatrix} \begin{bmatrix} R_{E_x} \\ L_{E_x} \\ R_{E_y} \\ L_{E_y} \end{bmatrix} = (T_B) \begin{bmatrix} R_{E_x} \\ L_{E_x} \\ R_{E_y} \\ L_{E_y} \end{bmatrix}$$

and

$$\begin{bmatrix} R_{E_x} \\ L_{E_x} \\ R_{E_y} \\ L_{E_y} \end{bmatrix} = \begin{bmatrix} \cos \phi & 0 & -\sin \phi & 0 \\ 0 & \cos \phi & 0 & -\sin \phi \\ \sin \phi & 0 & \cos \phi & 0 \\ 0 & \sin \phi & 0 & \cos \phi \end{bmatrix} \begin{bmatrix} R_{E_{x'}} \\ L_{E_{x'}} \\ R_{E_{y'}} \\ L_{E_{y'}} \end{bmatrix} \quad (35)$$

Figure 5 is an extension of Figure 4 with one plate added, so that the overall transmission can be written from Equation 32 as

$$\begin{bmatrix} R_{E4x'} \\ L_{E4x'} \\ R_{E4y'} \\ L_{E4y'} \end{bmatrix} = (T_{B2}) (T_B) (T_f)^{-1} (T_f) (T_f) (T_{B1}) \begin{bmatrix} R_{E1x} \\ L_{E1x} \\ R_{E1y} \\ L_{E1y} \end{bmatrix}$$

and

$$\begin{bmatrix} R_{E4x} \\ L_{E4x} \\ R_{E4y} \\ L_{E4y} \end{bmatrix} = (T_B)^{-1} (T_{B2}) (T_B) (T_f)^{-1} (T_f) (T_f) (T_{B1}) \begin{bmatrix} R_{E1x} \\ L_{E1x} \\ R_{E1y} \\ L_{E1y} \end{bmatrix} \quad (36)$$

Possibly a word of caution is in order here. Suppose there are three birefringent slabs in a system with axes, xy , $x'y'$, and $x''y''$, rotated at angles θ , ϕ , and θ to the xy axes. A proper set of matrices for this system is then

$$(T_B(\theta))^{-1} (T_{B3}) (T_B(\theta-\phi)) (T_{B2}) (T_B(\phi)) (T_{B1}) = \begin{bmatrix} \cos \theta & 0 & -\sin \theta & 0 \\ 0 & \cos \theta & 0 & -\sin \theta \\ \sin \theta & 0 & \cos \theta & 0 \\ 0 & \sin \theta & 0 & \cos \theta \end{bmatrix} (T_{B3})$$

$$\begin{bmatrix} \cos(\theta-\phi) & 0 & \sin(\theta-\phi) & 0 \\ 0 & \cos(\theta-\phi) & 0 & \sin(\theta-\phi) \\ -\sin(\theta-\phi) & 0 & \cos(\theta-\phi) & 0 \\ 0 & -\sin(\theta-\phi) & 0 & \cos(\theta-\phi) \end{bmatrix} (T_{B2}) \begin{bmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & \cos \phi & 0 & \sin \phi \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & -\sin \phi & 0 & \cos \phi \end{bmatrix} (T_{B1}) \quad (37)$$

In order to end up in the original xy coordinate system the transformation ($T_B(\theta-\phi)$) is introduced since at that point it is necessary to transform from the $x'y'$ to $x''y''$ not from xy to $x''y''$.

PART D: A PARTICULAR EXAMPLE--THE FARADAY ROTATION ISOLATOR

Consider the isolator in Figure 6. The basic principle is, of course, the Faraday plate rotates the x -polarized wave 45 degrees so that it passes unattenuated through the second polarizer. A (reflected) wave traveling from the right is rotated 45 degrees further and ends up y polarized at the first polarizer and thus cannot be transmitted on to the left-hand side of the system. A little thought shows, assuming a lossless Faraday element, that at least one polarizer must absorb along its non-polar (y or y') axis; otherwise, the wave would be reflected from polarizer 1, rotated a further 45 degrees to be polarized along the y' axis, reflected, rotated 45 degrees to be polarized along the x axis, and passed by the first polarizer back to the generator side.

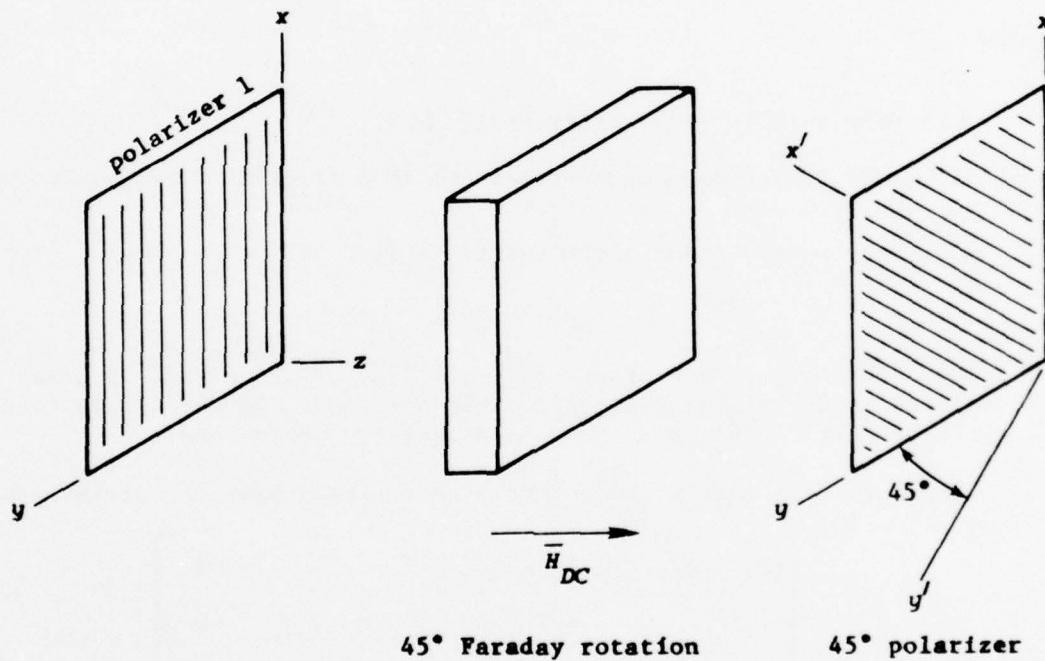


FIGURE 6. Faraday Rotation Isolator.

In microwave isolators, the polarizers are frequently rectangular TE_{10} waveguides which propagate only E_x (or $E_{x'}$) with E_y (or $E_{y'}$) below cutoff. A strip of absorber is placed either along the y direction on the input side and/or the y' direction on the output side of the Faraday element to avoid multiple reflections and loss of isolation.

Figure 6 is a special case of Figure 5 and the general transmission is given by Equation 36. As an example to illustrate a point we would like to assume perfect polarizers and Faraday element. It turns out that this is conceptually ambiguous even though it would simplify the mathematics.

Let the scattering matrix for the polarizer be given by:

$$\begin{bmatrix} L_{E_{1x}} \\ R_{E_{2x}} \\ L_{E_{1y}} \\ R_{E_{2y}} \end{bmatrix} = \begin{bmatrix} \delta_1 & F & 0 & 0 \\ F & \delta_1 & 0 & 0 \\ 0 & 0 & \delta_3 & \delta_2 \\ 0 & 0 & \delta_2 & \delta_3 \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{2x}} \\ R_{E_{1y}} \\ L_{E_{2y}} \end{bmatrix} \quad (38)$$

where

$|\delta_1|$ (the x -reflection coefficient) $\ll 1$.

$|F|$ (the x -transmission coefficient) is a fraction very nearly equal to 1.

$|\delta_3|$ (the y -reflection coefficient) $\ll 1$.

$|\delta_2|$ (the y -transmission coefficient) $\ll 1$.

These conditions mean practically all x -polarized signal is transmitted and the y -polarized signal mostly absorbed. As the δ 's approach zero the conditions for a perfect polarizer are approached.

The scattering matrix is converted to the transmission matrix by use of Equation 10:

$$(T_{B1}) = \begin{bmatrix} (F^2 - \delta_1^2)/F & \delta_1/F & 0 & 0 \\ -\delta_1/F & 1/F & 0 & 0 \\ 0 & 0 & (\delta_2^2 - \delta_3^2)/\delta_2 & \delta_3/\delta_2 \\ 0 & 0 & -\delta_3/\delta_2 & 1/\delta_2 \end{bmatrix} \quad (39)$$

Equation 39 shows why it is difficult to work with the scattering matrix for a perfect polarizer, since t_{33} , t_{34} , t_{43} become indeterminant,

while t_{44} goes to infinity. It need not surprise us that some coefficients are greater than one in magnitude since (referring to Equation 7) the relationship, for example, between R_{E2y} to R_{E1y} is not output to input, as somewhat implied, but rather input to output.

Turning now to Equation 18 and (T_F), it is not possible to produce an ideal non-absorbing Faraday plate. This requires $\gamma_{\pm} = \alpha_{\pm} + jB_{\pm} = jB_{\pm}$. It is furthermore required that $r_{\pm} = 0$. This in turn indicates $r_+ = r_-$ and therefore $B_+ = B_-$, which in turn means zero rotation per unit length.

By letting $\epsilon_{\pm} = \epsilon(1 \pm \delta)$ there results

$$r_{\pm} = \frac{1 - \sqrt{\frac{\epsilon}{\mu}(1 \pm \delta)}}{1 + \sqrt{\frac{\epsilon}{\mu}(1 \pm \delta)}} \quad (40)$$

Choosing $\epsilon = \mu$ and making δ arbitrarily small (by, for example, reducing the applied DC magnetic field) we arrive at

$$r_{\pm} \approx \pm \delta/4 \quad (41)$$

The rotation is given by¹⁸

$$\theta = \frac{1}{2} [\text{Phase } R_{E2-} - \text{Phase } R_{E2+}] \quad (42)$$

For small r_{\pm} this is essentially

$$\theta = \frac{1}{2} (B_- d - B_+ d) \quad (43)$$

Since

$$B_{\pm} = \frac{\omega}{c} \sqrt{\mu \epsilon_{\pm}} = \frac{\omega}{c} \sqrt{\mu \epsilon} (1 \pm \frac{\delta}{2}) \quad (44)$$

then

$$\theta = \frac{\delta \omega d}{2c} \sqrt{\mu \epsilon} \quad (45)$$

where d is the thickness of the Faraday plate. Thus, by making d arbitrarily large we can in principle obtain 45 degree rotation while having $r_{\pm} \rightarrow 0$ for small δ . Since $\alpha_{\pm} = 0$, the ellipticity of the transmitted wave also goes to zero.

For an isolator

$$\theta = \frac{1}{2} (\phi_+ - \phi_-) = \frac{\pi}{4} \quad (46)$$

Thus the scattering matrix for an ideal Faraday plate is

$$(S_F) = \begin{bmatrix} & -j\phi_+ & 0 & 0 \\ 0 & e^{-j\phi_+} & 0 & 0 \\ e^{-j\phi_+} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-j\phi_-} \\ 0 & 0 & e^{-j\phi_-} & 0 \end{bmatrix} \quad (47)$$

and the transmission matrix

$$(T_F) = \begin{bmatrix} -j\phi_+ & 0 & 0 & 0 \\ e^{-j\phi_+} & 0 & 0 & 0 \\ 0 & e^{j\phi_+} & 0 & 0 \\ 0 & 0 & -j\phi_- & 0 \\ 0 & 0 & 0 & e^{j\phi_-} \end{bmatrix} \quad (48)$$

From Equation 46, this may be rewritten

$$(T_F) = \begin{bmatrix} -je^{-j\phi} & 0 & 0 & 0 \\ 0 & je^{j\phi} & 0 & 0 \\ 0 & 0 & e^{-j\phi} & 0 \\ 0 & 0 & 0 & e^{j\phi} \end{bmatrix} \quad (49)$$

where $\phi = \phi_-$.

Utilizing this matrix causes no particular difficulty in our isolator analysis.

For the 45-degree polarizer the coordinate rotation matrix (T_B) is

$$(T_B) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad (50)$$

and the birefringent plate matrix (T_{B2}) is identical to (T_{B1}) assuming identical polarizers.

It is simple if tedious to multiply out Equation 36 assuming Equation 50 to yield

$$\begin{bmatrix} R_{E_{4x'}} \\ L_{E_{4x'}} \\ R_{E_{4y'}} \\ L_{E_{4y'}} \end{bmatrix} = (T_I) \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (51)$$

where

$$\begin{aligned} 2t_{I11} &= (t_{f11}t_{11}^2 - 2t_{f12}t_{11}t_{12} - t_{f22}t_{12}^2) e^{j\pi/4} \\ &\quad + (t_{f33}t_{11}^2 - 2t_{f34}t_{11}t_{12} - t_{f44}t_{12}^2) e^{-j\pi/4} \\ 2t_{I12} &= (t_{f11}t_{12}t_{11} + t_{f22}t_{22}t_{12} + t_{f12}t_{11}t_{22} - t_{f12}t_{12}^2) e^{j\pi/4} \\ &\quad + (t_{f33}t_{11}t_{12} + t_{f34}t_{11}t_{22} + t_{f44}t_{12}t_{22} - t_{f34}t_{12}^2) e^{-j\pi/4} \\ 2t_{I13} &= (t_{f33}t_{11}t_{33} - t_{f34}t_{11}t_{34} - t_{f34}t_{12}t_{33} - t_{f44}t_{12}t_{34}) e^{j\pi/4} \\ &\quad + (t_{f11}t_{11}t_{33} - t_{f12}t_{11}t_{34} - t_{f12}t_{12}t_{33} - t_{f22}t_{12}t_{34}) e^{-j\pi/4} \\ 2t_{I14} &= (t_{f34}t_{11}t_{44} + t_{f33}t_{11}t_{34} + t_{f44}t_{12}t_{44} - t_{f34}t_{12}t_{34}) e^{j\pi/4} \\ &\quad + (t_{f12}t_{11}t_{44} + t_{f11}t_{11}t_{34} + t_{f22}t_{12}t_{44} - t_{f12}t_{12}t_{34}) e^{-j\pi/4} \end{aligned}$$

$$\begin{aligned}
{}^2t_{121} &= (t_{f12}t_{12}^2 - t_{f11}t_{11}t_{12} - t_{f12}t_{11}t_{22} - t_{f22}t_{12}t_{22}) e^{j\pi/4} \\
&\quad + (t_{f34}t_{12}^2 - t_{f33}t_{11}t_{12} - t_{f34}t_{11}t_{22} - t_{f44}t_{12}t_{22}) e^{-j\pi/4} \\
{}^2t_{122} &= (t_{f22}t_{22}^2 - 2t_{f12}t_{12}t_{22} - t_{f11}t_{12}^2) e^{j\pi/4} \\
&\quad + (t_{f44}t_{22}^2 - 2t_{f34}t_{12}t_{22} - t_{f33}t_{12}^2) e^{-j\pi/4} \\
{}^2t_{123} &= (t_{f34}t_{12}t_{34} - t_{f33}t_{12}t_{33} - t_{f34}t_{22}t_{33} - t_{f44}t_{22}t_{34}) e^{j\pi/4} \\
&\quad + (t_{f12}t_{12}t_{34} - t_{f11}t_{12}t_{33} - t_{f12}t_{22}t_{33} - t_{f22}t_{22}t_{34}) e^{-j\pi/4} \\
{}^2t_{124} &= (t_{f44}t_{22}t_{44} - t_{f34}t_{22}t_{34} - t_{f34}t_{12}t_{44} - t_{f33}t_{12}t_{34}) e^{j\pi/4} \\
&\quad + (t_{f22}t_{22}t_{44} - t_{f12}t_{22}t_{34} - t_{f12}t_{12}t_{44} - t_{f11}t_{12}t_{34}) e^{-j\pi/4} \\
{}^2t_{131} &= (t_{f34}t_{12}t_{33} - t_{f33}t_{11}t_{33} + t_{f44}t_{12}t_{34} + t_{f34}t_{11}t_{34}) e^{j\pi/4} \\
&\quad + (t_{f12}t_{12}t_{33} - t_{f11}t_{11}t_{33} + t_{f22}t_{12}t_{34} + t_{f12}t_{11}t_{34}) e^{-j\pi/4} \\
{}^2t_{132} &= (t_{f34}t_{12}t_{34} - t_{f44}t_{22}t_{34} - t_{f33}t_{12}t_{33} - t_{f34}t_{22}t_{33}) e^{j\pi/4} \\
&\quad + (t_{f12}t_{12}t_{34} - t_{f22}t_{22}t_{34} - t_{f11}t_{12}t_{33} - t_{f12}t_{22}t_{33}) e^{-j\pi/4} \\
{}^2t_{133} &= (t_{f11}t_{33}^2 - 2t_{f12}t_{33}t_{34} - t_{f22}t_{34}^2) e^{j\pi/4} \\
&\quad + (t_{f33}t_{33}^2 - 2t_{f34}t_{33}t_{34} - t_{f44}t_{34}^2) e^{-j\pi/4} \\
{}^2t_{134} &= (t_{f11}t_{33}t_{34} + t_{f12}t_{33}t_{44} + t_{f22}t_{34}t_{44} - t_{f12}t_{34}^2) e^{j\pi/4} \\
&\quad + (t_{f33}t_{33}t_{34} + t_{f34}t_{33}t_{44} + t_{f44}t_{34}t_{44} - t_{f34}t_{34}^2) e^{-j\pi/4} \\
{}^2t_{141} &= (t_{f44}t_{12}t_{44} + t_{f34}t_{11}t_{44} - t_{f34}t_{12}t_{34} + t_{f33}t_{11}t_{34}) e^{j\pi/4} \\
&\quad + (t_{f22}t_{12}t_{44} + t_{f12}t_{11}t_{44} - t_{f12}t_{12}t_{34} + t_{f11}t_{11}t_{34}) e^{-j\pi/4} \\
{}^2t_{142} &= (t_{f33}t_{12}t_{34} + t_{f34}t_{22}t_{34} + t_{f34}t_{12}t_{44} - t_{f44}t_{22}t_{44}) e^{j\pi/4} \\
&\quad + (t_{f11}t_{12}t_{34} + t_{f12}t_{22}t_{34} + t_{f12}t_{12}t_{44} - t_{f22}t_{22}t_{44}) e^{-j\pi/4}
\end{aligned}$$

$$\begin{aligned}
 {}^2t_{143} &= (t_{f12}t_{34}^2 - t_{f11}t_{33}t_{34} - t_{f12}t_{33}t_{44} - t_{f22}t_{34}t_{44}) e^{j\pi/4} \\
 &\quad + (t_{f34}t_{34}^2 - t_{f33}t_{33}t_{34} - t_{f44}t_{34}t_{44} - t_{f34}t_{33}t_{44}) e^{-j\pi/4} \\
 {}^2t_{144} &= (t_{f22}t_{44}^2 - 2t_{f12}t_{34}t_{44} - t_{f11}t_{34}^2) e^{j\pi/4} \\
 &\quad + (t_{f44}t_{44}^2 - 2t_{f34}t_{34}t_{44} - t_{f33}t_{34}^2) e^{-j\pi/4} \tag{52}*
 \end{aligned}$$

When Equation 49 is plugged into this rather horrendous set of equations, things simplify considerably to

$$\begin{aligned}
 t_{111} &= t_{11}^2 e^{-j\phi} e^{-j\pi/4} \\
 t_{112} &= t_{11}t_{12} e^{-j\phi} e^{-j\pi/4} \\
 t_{113} &= -t_{12}t_{34} e^{j\phi} e^{j\pi/4} \\
 t_{114} &= t_{12}t_{44} e^{j\phi} e^{j\pi/4} \\
 t_{121} &= -t_{11}t_{12} e^{-j\phi} e^{-j\pi/4} \\
 t_{122} &= -t_{12}^2 e^{-j\phi} e^{-j\pi/4} \\
 t_{123} &= -t_{22}t_{34} e^{j\phi} e^{j\pi/4} \\
 t_{124} &= t_{22}t_{44} e^{j\phi} e^{j\pi/4} \\
 t_{131} &= t_{12}t_{34} e^{j\phi} e^{j\pi/4} \\
 t_{132} &= -t_{22}t_{34} e^{j\phi} e^{j\pi/4} \\
 t_{133} &= t_{33}^2 e^{-j\phi} e^{-j\pi/4}
 \end{aligned}$$

* Equation 52 can be directly checked by using the equation:

$$t_{Iij} = (\mathbb{T}_B)_{ik} (\mathbb{T}_B)_{kl} (\mathbb{T}_F)_{lm}^{-1} (\mathbb{T}_F)_{mn} (\mathbb{T}_F)_{np} (\mathbb{T}_B)_{pj}$$

where repeated indices indicate a sum.

$$\begin{aligned}
 t_{134} &= t_{33} t_{34} e^{-j\phi} e^{-j\pi/4} \\
 t_{141} &= t_{12} t_{44} e^{j\phi} e^{j\pi/4} \\
 t_{142} &= -t_{22} t_{44} e^{+j\phi} e^{j\pi/4} \\
 t_{143} &= -t_{33} t_{34} e^{-j\phi} e^{-j\pi/4} \\
 t_{144} &= -t_{34}^2 e^{-j\phi} e^{-j\pi/4}
 \end{aligned} \tag{53}$$

While all 16 components of the matrix are present, Equation 53 is a large improvement over Equation 52. Substituting Equation 39 into Equation 53 results in

$$t_{111} \approx F^2 e^{-j\phi} e^{-j\pi/4}$$

$$t_{112} \approx \delta_1 e^{-j\phi} e^{-j\pi/4}$$

$$t_{113} = \frac{-\delta_1 \delta_3}{\delta_2 F} e^{j\phi} e^{j\pi/4}$$

$$t_{114} = \frac{\delta_1}{\delta_2 F} e^{j\phi} e^{j\pi/4}$$

$$t_{121} \approx -\delta_1 e^{-j\phi} e^{-j\pi/4}$$

$$t_{122} \approx 0$$

$$t_{123} = \frac{-\delta_3}{\delta_2 F} e^{j\phi} e^{j\pi/4}$$

$$t_{124} = \frac{1}{\delta_2 F} e^{j\phi} e^{j\pi/4}$$

$$t_{131} = \frac{\delta_1 \delta_3}{\delta_2 F} e^{j\phi} e^{j\pi/4}$$

$$t_{132} = \frac{-\delta_3}{\delta_2 F} e^{j\phi} e^{j\pi/4}$$

$$t_{133} = \frac{\delta_3^4}{\delta_2^2} e^{-j\phi} e^{-j\pi/4}$$

$$t_{134} = \delta_3 \left(1 - \frac{\delta_3^2}{\delta_2^2} \right) e^{-j\phi} e^{-j\pi/4}$$

$$t_{141} = \frac{\delta_1}{\delta_2^F} e^{j\phi} e^{j\pi/4}$$

$$t_{142} = \frac{-1}{\delta_2^F} e^{j\phi} e^{j\pi/4}$$

$$t_{143} = \delta_3 \left(\frac{\delta_3^2}{\delta_2^2} - 1 \right) e^{-j\phi} e^{-j\pi/4}$$

$$t_{144} = \frac{-\delta_3^2}{\delta_2^2} e^{-j\phi} e^{-j\pi/4} \quad (54)$$

where some of the smaller terms have been dropped. Some of the terms in Equation 54 are larger than one in absolute value, and some are indeterminate depending on the relative values of δ_1 , δ_2 , and δ_3 . Again, as in Equation 39 it should be borne in mind that inputs (waves traveling toward the isolator) and outputs (waves traveling away from the isolator) appear on both sides of Equation 36.

A clearer picture arises if Equation 9 is used to transform Equation 54 back into the scattering matrix, or

$$S_{11} \approx \delta_1$$

$$S_{12} \approx 0$$

$$S_{13} \approx 0$$

$$S_{14} \approx -\delta_2^F e^{-j\phi} e^{-j\pi/4}$$

$$S_{21} \approx F^2 e^{-j\phi} e^{-j\pi/4}$$

$$\begin{aligned}
s_{22} &\approx \delta_1 \\
s_{23} &\approx 0 \\
s_{24} &\approx 0 \\
s_{31} &\approx 0 \\
s_{32} &\approx \delta_2^F e^{-j\phi} e^{-j\pi/4} \\
s_{33} &\approx \delta_3 \\
s_{34} &\approx 0 \\
s_{41} &\approx 0 \\
s_{42} &\approx 0 \\
s_{43} &\approx 0 \\
s_{44} &\approx \delta_3
\end{aligned} \tag{55}$$

where terms of order greater than δ^1 have been dropped. Written out in matrix notation this is

$$\begin{bmatrix} L_{E_{1x}} \\ R_{E_{4x'}} \\ L_{E_{1y}} \\ R_{E_{4y'}} \end{bmatrix} = \begin{bmatrix} \delta_1 & 0 & 0 & -\delta_2^F e^{-j\phi} e^{-j\pi/4} \\ F^2 e^{-j\phi} e^{-j\pi/4} & \delta_1 & 0 & 0 \\ 0 & \delta_2^F e^{-j\phi} e^{-j\pi/4} & \delta_3 & 0 \\ 0 & 0 & 0 & \delta_3 \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{4x'}} \\ R_{E_{1y}} \\ L_{E_{4y'}} \end{bmatrix} \tag{56}$$

This scattering matrix is not symmetrical even though each of the components making up the isolator has a symmetrical scattering matrix. This is evidently because the polarizers are not lossless--specifically they have a very heavy absorption along the y axis, this being true even if $F \rightarrow 1$, $\delta_1, \delta_2, \delta_3 \rightarrow 0$.

An examination of Equation 55, Figure 6, and Equation 38 allows the matrix elements to be deduced directly, as follows:

- s_{11} : The input x polarized wave from the left is reflected directly as δ_1 . The next higher order x term from this component is transmitted, (F), rotated, reflected ($\delta_1 F$) rotated to the y plane, reflected ($\delta_1 \delta_2 F$), etc., so $s_{11} \approx \delta_1$.
- s_{12} : The reflected x polarized wave on the left is related to the left traveling x' polarized input wave from the right by going through the right polarizer (F), being rotated to the y direction, being reflected ($\delta_3 F$) being rotated to the y' direction, reflected ($\delta_3^2 F$), etc., so $s_{12} \approx 0$.
- s_{13} : The reflected x polarized wave on the left is related to the right traveling y polarized input by transmission (δ_2), rotation to the y' direction, reflection ($\delta_2 \delta_3$), so $s_{13} \approx 0$.
- s_{14} : The reflected x polarized wave on the left is related to the y' polarized input wave from the right by transmission (δ_2) rotation to the (minus) x direction and transmission ($-\delta_2 F$) times the phase retardation of the Faraday plate, $e^{-j\phi} e^{-j\pi/4}$. Therefore, $s_{14} \approx -\delta_2 F e^{-j\phi} e^{-j\pi/4}$.
- s_{21} : The right traveling x' polarized component is related to the right traveling x polarized component by transmission F , rotation and retardation $e^{-j\phi} e^{-j\pi/4}$, and transmission F . Thus, $s_{21} = F^2 e^{-j\phi} e^{-j\pi/4}$.
- s_{22} : The right (RT) x' polarized component is related to the left (LT) x' component by reflection, δ_1 , all other terms being of higher order. $s_{22} \approx \delta_1$.
- s_{23} : The RT x' polarized component is related to the RT y polarized component by transmission through the first polarizer δ_2 , rotation to the y' direction, reflection, $\delta_3 \delta_2$, rotation to the x direction, reflection $\delta_1 \delta_3 \delta_2$, rotation to the x' direction, and transmission $F \delta_1 \delta_3 \delta_2 \approx 0$.
- s_{24} : The RT x' polarized component is related to the LT y' component by transmission δ_2 , rotation to the x direction, reflection $\delta_1 \delta_2$, rotation to the x' direction and transmission $F \delta_1 \delta_2 \approx 0$.
- s_{31} : The LT y component is related to the RT x component by transmission, F , rotation to the x' direction, reflection $\delta_1 F$, rotation to the y direction, and transmission $\delta_2 \delta_1 F \approx 0$.
- s_{32} : The LT y polarized wave is related to the LT x' polarized wave by transmission, F , rotation to the y direction (times the phase retardation $e^{-j\phi} e^{-j\pi/4}$) $F e^{-j\phi} e^{-j\pi/4}$, and transmission. $s_{32} = F \delta_2 e^{-j\phi} e^{-j\pi/4}$.

- S_{33} : The (LT) y polarized wave is related to the RT y wave by reflection δ_3 , all higher order terms being negligible.
 $S_{33} = \delta_3$.
- S_{34} : The LT y component is related to the LT y' component by transmission δ_2 , rotation to the x direction, reflection $\delta_1\delta_2$, etc.
 $S_{34} = 0$.
- S_{41} : The RT y' component is related to the RT x component by transmission F , rotation to the x' direction, reflection δ_1F , rotation to the y direction, reflection $\delta_3\delta_1F$, rotation to the y' direction, and transmission (ignoring the phase retardations) or $\delta_2\delta_3\delta_1F \approx 0$.
- S_{42} : The RT y' component is related to the LT x' component by transmission F , rotation to the y direction, reflection δ_3F , rotation to the y' direction, and transmission $\delta_2\delta_3F \approx 0$.*
- S_{43} : The RT y' component is related to the RT y component by transmission δ_2 , rotation to the y' plane, and transmission $\delta_2^2 \approx 0$.
- S_{44} : The RT y' component is related to the LT y' component by reflection δ_3 .

The point of the exercise, however, is not so much to show a complicated derivation for what can be arrived at with a little physical reasoning, but merely to demonstrate that the approach outlined in this report converges to the proper answer.

* These scattering coefficients can be summed to infinity since there is no reflection at the Faraday plate, so that

$$\begin{aligned} S_{42} &= \delta_2 \delta_3^F e^{-j2\phi} e^{-j\pi/2} + \delta_2^2 \delta_1^3 F e^{-j4\phi} e^{-j\pi} + \delta_2^4 \delta_1^5 F e^{-j6\phi} e^{-j3\pi/2} + \dots \\ &= \delta_2 \delta_3^F e^{-j2\phi} e^{-j\pi/2} \sum_{N=0}^{\infty} (\delta_1 \delta_3)^{2N} e^{-j2N\phi} e^{-jN\pi/2} \\ &= \frac{\delta_2 \delta_3^F e^{-j2\phi} e^{-j\pi/2}}{1 - \delta_1^2 \delta_3^2 e^{-j2\phi} e^{-j\pi/2}} \end{aligned}$$

For perfect polarizers, $F = 1$, $\delta_1 = \delta_2 = \delta_3 = 0$, and Equation 56 becomes just

$$\begin{bmatrix} L_{E_{1x}} \\ R_{E_{4x'}} \\ L_{E_{1y}} \\ R_{E_{4y'}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ e^{-j\phi} e^{-j\pi/4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{4x'}} \\ R_{E_{1y}} \\ L_{E_{4y'}} \end{bmatrix} \quad (57)$$

Equation 57 can be compared to Equation 2, and the relationship is obvious. Since the matrix is 4×4 rather than 2×2 , however, perhaps it is clearer that the scattering matrix does not represent a truly lossless system and therefore is not necessarily symmetrical (Ref. 27). A "Quasi-Isolator," formed by a polarizer followed by a quarter-wave plate is discussed in Appendix A.

III. CONCLUSION

We have outlined a method for determining the overall transmission and reflection coefficients at normal incidence, for an "N" layered system, where each layer consists of either a Faraday plate or birefringent plate with the Faraday or a principal axis, respectively, lying along the direction of propagation.

Another example of the application of this method is worked out in Appendix B. Considered is the rotary vane phase shifter, widely used in microwave and millimeter wave circuitry.

The case where a layer shows both birefringence and Faraday rotation is not included in this analysis, although it appears to be a straightforward extension. In this case the normal modes of the plate would be \pm elliptically polarized (Ref. 19, 23) and so the transformation would be from linear to \pm elliptical form rather than \pm circular.

The more general case where the individual plates show Faraday rotation and birefringence while propagation is not necessarily along either a principal or Faraday axis can probably also be treated by an extension of the methods given here, although the details have not been carried out. In this case Equation 15 would contain all nine terms with $\epsilon_{ij} \neq \epsilon_{ji}$. The solution consists of \pm elliptically polarized waves for the RF H fields normal to the direction of propagation. The exact form of A_+ , A_- , γ_+ , and γ_- in the solution

$$\bar{H} = H_1 e^{-\gamma_+ z} (\hat{x} + A_+ \hat{y}) + H_2 e^{-\gamma_- z} (\hat{x} + A_- \hat{y}) \quad (58)$$

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is determined by ρ_{xx} , ρ_{xy} , ρ_{yx} , ρ_{yy} where ρ_{ij} are the elements of $(\epsilon)^{-1}$, as shown by Penz (Ref. 19).

The accompanying $\bar{E}_{1,2}$ vectors possess an electric field component in the direction of propagation so that the Poynting vector $\bar{E} \times \bar{H}$ lies at some angle to the direction of propagation (\hat{z}).

Nonetheless, although the necessary foot work has not been carried out, it appears intuitively feasible to extend the analysis presented here to this general case, if such appeared desirable.

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Appendix A

A QUASI-ISOLATOR FORMED BY A POLARIZER FOLLOWED BY A QUARTER-WAVE PLATE

Consider Figure A-1. The polarizer scattering matrix is given by Equation 38 and the transmission matrix by Equation 39. The rotation through 45 degrees from the xy coordinates to the $x'y'$ coordinate system is given by Equation 50.

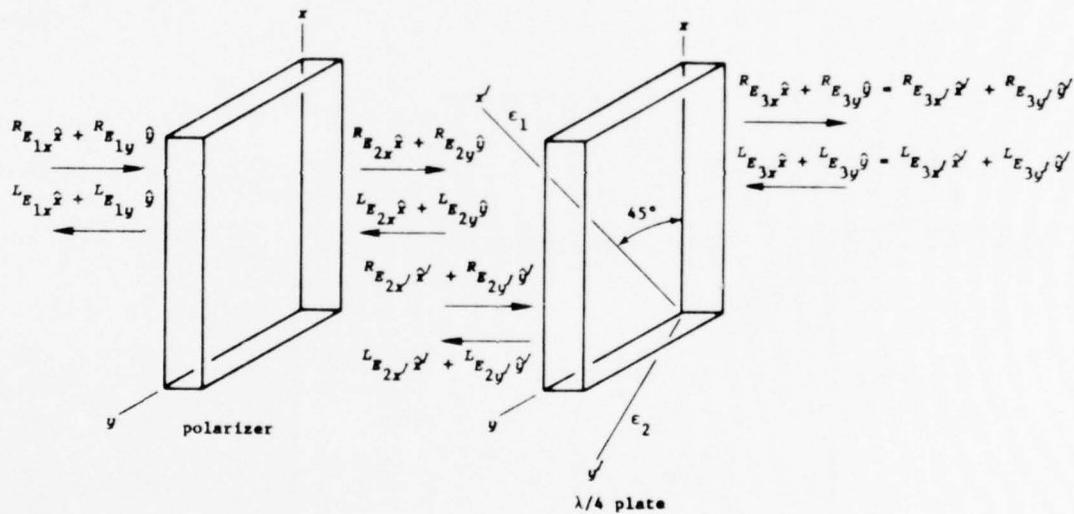


FIGURE A-1. Isolator Formed by Polarizer Followed by a Quarter-Wave Plate.

The form of the scattering matrix for the quarter-wave (birefringent) plate is given by Equation 6. In optical work surface reflection coefficients are frequently ignored in discussing $\lambda/4$ and $\lambda/2$ plates. In any case, assuming a lossless medium where (for example) $\epsilon_1 = 1$ and $\epsilon_2 = 1 + \Delta$,

by making the path longer and longer and Δ smaller and smaller we can make the reflections at boundaries smaller and smaller and thus approach arbitrarily close to $S_{11} = S_{33} = 0$. For a perfect $\lambda/4$ plate the scattering matrix is

$$(S) = \begin{bmatrix} 0 & S_{12} & 0 & 0 \\ S_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{34} \\ 0 & 0 & S_{34} & 0 \end{bmatrix} \quad (\text{A-1})$$

Taking the $\lambda/4$ plate as lossless, Equation 5 reduces to

$$\begin{aligned} S_{12} &= e^{-jB_x' d} \\ S_{34} &= e^{-jB_y' d} \end{aligned} \quad (\text{A-2})$$

The definition of a $\lambda/4$ plate requires

$$(B_x' - B_y')d = \pm \frac{\pi}{2} \quad (\text{A-3})$$

From Equation 10, the transmission matrix is just

$$(T_2) = \begin{bmatrix} e^{-jB_x' d} & 0 & 0 & 0 \\ 0 & e^{jB_x' d} & 0 & 0 \\ 0 & 0 & e^{-jB_y' d} & 0 \\ 0 & 0 & 0 & e^{jB_y' d} \end{bmatrix} \quad (\text{A-4})$$

At this point we have all that is required to go from $R_{E_{1x}}$, $R_{E_{1y}}$, $L_{E_{1x}}$, $L_{E_{1y}}$, to $R_{E_{3x'}}$, $R_{E_{3y'}}$, $L_{E_{3x'}}$ and $L_{E_{3y'}}$ in Figure A-1. Before carrying out the required matrix multiplication Equation A-3 can be used to rewrite Equation A-4 as

$$(T_2) = \begin{bmatrix} -jB_{x'}d & 0 & 0 & 0 \\ 0 & e^{jB_{x'}d} & 0 & 0 \\ 0 & 0 & \pm j e^{-jB_{x'}d} & 0 \\ 0 & 0 & 0 & \mp j e^{jB_{x'}d} \end{bmatrix} \quad (A-5)$$

Now

$$\begin{bmatrix} R_{E_{3x'}} \\ L_{E_{3x'}} \\ R_{E_{3y'}} \\ L_{E_{3y'}} \end{bmatrix} = (T_2) (T_B) (T_1) \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (A-6)$$

where (T_B) is given by Equation 50 and (T_1) by Equation 39. Carrying out the indicated multiplication

$$\begin{bmatrix} R_{E_{3x'}} \\ L_{E_{3x'}} \\ R_{E_{3y'}} \\ L_{E_{3y'}} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{F^2 - \delta_1^2}{F} e^{-jB_{x'}d} & \frac{\delta_1}{F} e^{-jB_{x'}d} & \frac{\delta_2^2 - \delta_3^2}{\delta_2} e^{-jB_{x'}d} & \frac{\delta_3}{\delta_2} e^{-jB_{x'}d} \\ \frac{-\delta_1}{F} e^{-jB_{x'}d} & \frac{1}{F} e^{jB_{x'}d} & \frac{-\delta_3}{\delta_2} e^{jB_{x'}d} & \frac{1}{\delta_2} e^{jB_{x'}d} \\ \pm j \frac{\delta_1^2 - F^2}{F} e^{-jB_{x'}d} & \mp j \frac{\delta_1}{F} e^{-jB_{x'}d} & \pm j \frac{\delta_2^2 - \delta_3^2}{\delta_2} e^{-jB_{x'}d} & \pm j \frac{\delta_3}{\delta_2} e^{-jB_{x'}d} \\ \mp j \frac{\delta_1}{F} e^{jB_{x'}d} & \pm j \frac{1}{F} e^{jB_{x'}d} & \pm j \frac{\delta_3}{\delta_2} e^{jB_{x'}d} & \mp j \frac{1}{\delta_2} e^{jB_{x'}d} \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (A-7)$$

This can be inverted, using Equation 9 to find the scattering matrix, or

$$\begin{bmatrix} L_{E_{1x}} \\ R_{E_{3x'}} \\ L_{E_{1y}} \\ R_{E_{3y'}} \end{bmatrix} = \begin{bmatrix} \delta_1 & \frac{F}{\sqrt{2}} e^{-jB_{x'}d} & 0 & \pm j \frac{F}{\sqrt{2}} e^{-jB_{x'}d} \\ \frac{F}{\sqrt{2}} e^{-jB_{x'}d} & \frac{1}{2} (\delta_1 + \delta_3) e^{-j2B_{x'}d} & \frac{\delta_2}{\sqrt{2}} e^{-jB_{x'}d} & \pm j \frac{1}{2} (\delta_1 + \delta_3) e^{-j2B_{x'}d} \\ 0 & \frac{\delta_2}{\sqrt{2}} e^{-jB_{x'}d} & \delta_3 & \pm j \frac{\delta_2}{\sqrt{2}} e^{-jB_{x'}d} \\ \mp j \frac{F}{\sqrt{2}} e^{-jB_{x'}d} & \pm j \frac{1}{2} (\delta_3 - \delta_1) e^{-j2B_{x'}d} & \pm j \frac{\delta_2}{\sqrt{2}} e^{-jB_{x'}d} & (\delta_1 - \delta_3) e^{-j2B_{x'}d} \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{3x'}} \\ R_{E_{1y}} \\ L_{E_{3y'}} \end{bmatrix} \quad (A-8)$$

In order to more clearly understand the device, it seems easier to use a perfect polarizer where $\delta_1 = \delta_2 = \delta_3 = 0$, $F = 1$ so that Equation A-8 becomes

$$\sqrt{2} \begin{bmatrix} L_{E_{1x}} \\ R_{E_{3x'}} \\ L_{E_{1y}} \\ R_{E_{3y'}} \end{bmatrix} = \begin{bmatrix} 0 & e^{-jB_{x'}d} & 0 & \pm j e^{-jB_{x'}d} \\ e^{-jB_{x'}d} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm j e^{-jB_{x'}d} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{3x'}} \\ R_{E_{1y}} \\ L_{E_{3y'}} \end{bmatrix} \quad (A-9)$$

Unlike Equation 57 for the Faraday rotation isolator, Equation A-9 is symmetrical and gives little hint of any isolation-like qualities. However, note that

$$R_{E_{3x'}} = e^{-jB_{x'}d} (\hat{x}' \mp j \hat{y}') R_{E_{1x}} / \sqrt{2} \quad (A-10)$$

This, as the output of a $\lambda/4$ plate should be, is circularly polarized. Also,

$$L_{E_{1x}} = \left[\left(L_{E_{3x'}} \mp j L_{E_{3y'}} \right) e^{-jB_{x'}d} \right] \sqrt{2} \quad (A-11)$$

If $L_{E_3'}$ is circularly polarized, so that

$$\bar{L}_{E_3'} = L_{E_3} (\hat{x}' \mp j\hat{y}') \quad (A-12)$$

then $L_{E_{1x}} = 0$ by substitution.

If $\bar{L}_{E_3'}$ is generated by reflection of $R_{E_3'}$ off any isotropic material with reflection coefficient R , then Equation A-12 will be the proper form for $\bar{L}_{E_3'}$ and isolation will indeed occur.

For perfect polarizer and $\lambda/4$ plate, one doesn't have to go through all of the multiplications to see how the device works. The polarizer insures an x -polarized input. From Equation A-5 it is evident that the output of the $\lambda/4$ plate is the circularly polarized signal

$$R_{E_{3x'}} \hat{x}' e^{-jB_{x'} d} \pm R_{E_{3y'}} \hat{y}' e^{-jB_{x'} d} = \frac{R_{E_{1x}}}{\sqrt{2}} e^{-jB_{x'} d} (\hat{x}' \mp j\hat{y}')$$

Upon reflection by any scalar coefficient the signal retains its form as

$$\frac{R_{E_{1x}}}{\sqrt{2}} e^{-jB_{x'} d} (\hat{x}' \mp j\hat{y}')$$

Transmission in the opposite direction is given by the inverse of Equation A-5 or

$$(\tau_2)^{-1} = \begin{bmatrix} e^{jB_{x'} d} & 0 & 0 & 0 \\ 0 & e^{-jB_{x'} d} & 0 & 0 \\ 0 & 0 & \mp j e^{jB_{x'} d} & 0 \\ 0 & 0 & 0 & \pm j e^{-jB_{x'} d} \end{bmatrix} \quad (A-13)$$

Thus the signal passed through the $\lambda/4$ plate in the opposite direction is just $L_{E_{2x'}} e^{-jB_{x'} d} \hat{x}' + L_{E_{2y'}} (\pm j e^{-jB_{x'} d}) \hat{y}' = R_{E_{1x}} \frac{R_{E_{1x}}}{\sqrt{2}} e^{-jB_{x'} d} (\hat{x}' + \hat{y}')$.

From Figure 5 this is a signal polarized along the y axis which is absorbed, as $R_{E_{1x}}^R e^{-j2B_x' d} \hat{y}$.

Note that if the polarizer reflected instead of absorbed (i.e., $\delta_2 = 1$ instead of zero) the signal would again go through the $\lambda/4$ plate, emerging now as

$$R_{E_{3x'}} e^{-jB_x' d} \hat{x}' \pm R_{E_{3y'}} j e^{-jB_x' d} \hat{y}' = R \frac{R_{E_{1x}}}{\sqrt{2}} e^{-j3B_x' d} (\hat{x}' \pm j\hat{y}')$$

--that is, circularly polarized in the direction opposite to that obtained as the first pass through the $\lambda/4$ plate. The signal would again

be reflected (as $R^2 \frac{R_{E_{1x}}}{\sqrt{2}} e^{-j3B_x' d} (\hat{x}' \pm j\hat{y}')$), transmitted through the

$\lambda/4$ plate as in Equation A-13, ending as

$$R^2 \frac{R_{E_{1x}}}{\sqrt{2}} e^{-j4B_x' d} (\hat{x}' - \hat{y}') = R^2 R_{E_{1x}} e^{-j4B_x' d} \hat{x}$$

polarized along the x axis. Thus it is clear without absorption in the system, this device will not function as an isolator, as was also the case for the Faraday rotation isolator.

Appendix B

THE ROTARY VANE PHASE SHIFTER

A rather clever device used as a phase shifter at microwave and millimeter wave frequencies consists essentially of a $\lambda/4$ plate followed by a $\lambda/2$ plate followed by a $\lambda/4$ plate. The principal axes of the $\lambda/4$ plates are identical while the principal axes of the $\lambda/2$ plate are rotated with respect to the axes of the $\lambda/4$ plates to vary the phase shift. The input signal is linearly polarized at 45 degrees to $\lambda/4$ plate axes. Figure B-1 shows the general arrangement.

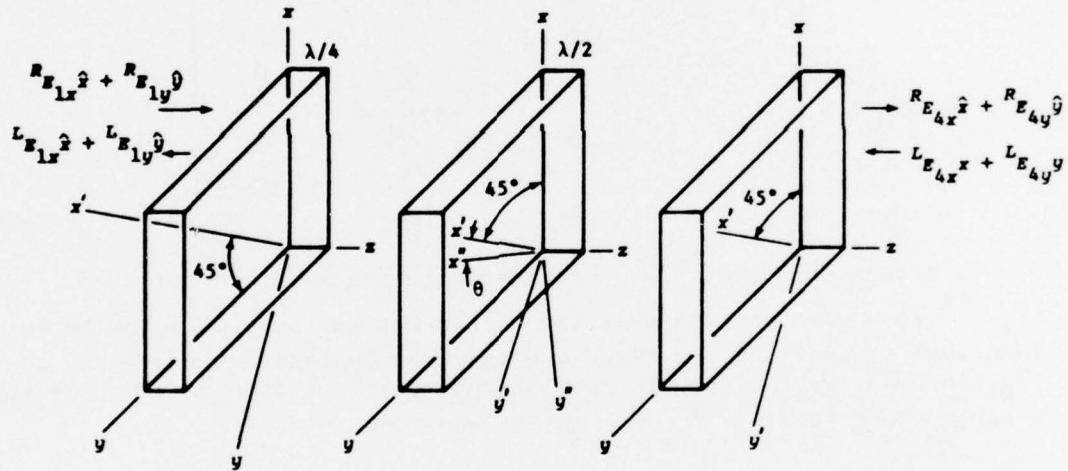


FIGURE B-1. The Rotary Vane Phase Shifter.

In waveguide a rectangular TE₁₀ waveguide supplies the proper polarization, there is transition from rectangular TE₁₀ to circular TE₁₁ where thin, fixed dielectric vanes at 45 degrees to the rectangular TE₁₀ *y* axis form the $\lambda/4$ plates. Between the $\lambda/4$ sections is another rotatable, (about the propagation direction or long axis of the waveguide) dielectric vane (twice as long as the $\lambda/4$ vanes) to form the $\lambda/2$ plate.

Just how the device works is not obvious (at least to the author) and the matrices form a handy way of analyzing the problem.

The transformation (T_{B1}) from the *x*,*y* to the *x'*,*y'* is given by Equation 50. The transmission matrix (T_{B1}) for a perfect $\lambda/4$ plate (no reflections) is given by Equation A-4, in general, or by Equation A-5 at the center frequency. The transformation (T_{B2}) from *x'*,*y'* axes to *x''*,*y''* axes is given by the first of Equation 35 with ϕ replaced by θ .

The transmission matrix (T_{B2}) for the $\lambda/2$ plate, being just twice as thick as the $\lambda/4$ plate is given by

$$(T_{B2}) = \begin{bmatrix} e^{-j2B_x' d} & 0 & 0 & 0 \\ 0 & e^{j2B_x' d} & 0 & 0 \\ 0 & 0 & e^{-j2B_y' d} & 0 \\ 0 & 0 & 0 & e^{j2B_y' d} \end{bmatrix} \quad (B-1)$$

To return from the *x''*,*y''* coordinates to the *x'*,*y'* coordinates, $(T_{B2})^{-1}$ is used. The transmission matrix for the second $\lambda/4$ plate is identical to that for the first and is given by Equation A-4 again as (T_{B1}) . Finally, the transformation from the *x'*,*y'* axes back to the *x*,*y* axes is given by $(T_{B1})^{-1}$. Our matrix equation is thus

$$\begin{bmatrix} R_{E4x} \\ L_{E4x} \\ R_{E4y} \\ L_{E4y} \end{bmatrix} = (T_{B1})^{-1} (T_{B1}) (T_{B2})^{-1} (T_{B2}) (T_{B2}) (T_{B1}) (T_{B1}) \begin{bmatrix} R_{E1x} \\ L_{E1x} \\ R_{E1y} \\ L_{E1y} \end{bmatrix} = (T_t) \begin{bmatrix} R_{E1x} \\ L_{E1x} \\ R_{E1y} \\ L_{E1y} \end{bmatrix} \quad (B-2)$$

This series of multiplications can be carried out fairly quickly by use of a chain rule:

$$(T_t)_{ij} = (T_{B1})_{ik}^{-1} (T_{B1})_{kl} (T_{B2})_{lm}^{-1} (T_{B2})_{mn} (T_{B2})_{np} (T_{B1})_{pg} (T_{B1})_{gj} \quad (B-3)$$

where repeated indices indicate a summation.

Although this is a rather long string of matrices, the simple form of the individual matrices enabled the procedure to be carried out reasonably readily, resulting in

$$\begin{aligned} 2t_{t11} &= \cos^2 \theta \left(e^{-j4B_x' d} + e^{-j4B_y' d} \right) + 2 \sin^2 \theta e^{-j2(B_x' + B_y')d} \\ &\quad + 2 \sin \theta \cos \theta \left(e^{-j(B_x' + 3B_y')d} - e^{-j(3B_x' + B_y')d} \right) \\ 2t_{t13} &= \cos^2 \theta \left(e^{-j4B_x' d} - e^{-j4B_y' d} \right) \\ 2t_{t22} &= \cos^2 \theta \left(e^{j4B_x' d} + e^{j4B_y' d} \right) + 2 \sin^2 \theta e^{j2(B_x' + B_y')d} \\ &\quad + 2 \sin \theta \cos \theta \left(e^{j(B_x' + 3B_y')d} - e^{j(3B_x' + B_y')d} \right) \\ 2t_{t24} &= \cos^2 \theta \left(e^{j4B_x' d} - e^{j4B_y' d} \right) \\ t_{t31} &= t_{t13} \\ 2t_{t33} &= \cos^2 \theta \left(e^{-j4B_x' d} - e^{-j4B_y' d} \right) + 2 \sin^2 \theta e^{-j2(B_x' + B_y')d} \\ &\quad + 2 \sin \theta \cos \theta \left(e^{-j(3B_x' + B_y')d} - e^{-j(B_x' + 3B_y')d} \right) \\ t_{t42} &= t_{t24} \\ 2t_{t44} &= \cos^2 \theta \left(e^{j4B_x' d} + e^{j4B_y' d} \right) + 2 \sin^2 \theta e^{j2(B_x' + B_y')d} \\ &\quad + 2 \sin \theta \cos \theta \left(e^{j(3B_x' + B_y')d} - e^{j(B_x' + 3B_y')d} \right) \\ t_{t12} &= t_{t14} = t_{t21} = t_{t23} = t_{t32} = t_{t34} = t_{t41} = t_{t43} = 0 \quad (B-4) \end{aligned}$$

When Equation A-3 applies, the transmission matrix reduces to

$$\begin{bmatrix} R_{E_{4x}} \\ L_{E_{4x}} \\ R_{E_{4y}} \\ L_{E_{4y}} \end{bmatrix} = \begin{bmatrix} e^{-j4B_{x'}d} e^{\mp j2\theta} & 0 & 0 & 0 \\ 0 & e^{j4B_{x'}d} e^{\pm j2\theta} & 0 & 0 \\ 0 & 0 & e^{-j4B_{x'}d} e^{\pm j2\theta} & 0 \\ 0 & 0 & 0 & e^{j4B_{x'}d} e^{\mp j2\theta} \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{1x}} \\ R_{E_{1y}} \\ L_{E_{1y}} \end{bmatrix} \quad (B-5)$$

or, in scattering matrix notation

$$\begin{bmatrix} L_{E_{1x}} \\ R_{E_{4x}} \\ L_{E_{1y}} \\ R_{E_{4y}} \end{bmatrix} = e^{-j4B_{x'}d} \begin{bmatrix} 0 & e^{\mp j2\theta} & 0 & 0 \\ e^{\mp j2\theta} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\pm j2\theta} \\ 0 & 0 & e^{\pm j2\theta} & 0 \end{bmatrix} \begin{bmatrix} R_{E_{1x}} \\ L_{E_{4x}} \\ R_{E_{1y}} \\ L_{E_{4y}} \end{bmatrix} \quad (B-6)$$

The input to such a device is usually a single component, such as $R_{E_{1x}}$ and the output would be $R_{E_{4x}}$. These are selected by polarizers such as the TE₁₀ mode rectangular waveguide for microwaves. The output $e^{-j4B_{x'}d}$ is thus retarded in phase by a fixed amount, $e^{-j4B_{x'}d}$ plus twice (\pm depending on whether the y' or x' axis is the slow or fast axis) the angular rotation of the $\lambda/2$ plate.

This approach does not add much to finding out just how the added phase shift $e^{\pm j2\theta}$ is obtained. This is perhaps most easily seen by breaking Equation B-2 into groups, or multiplying through one matrix at a time. Thus,

$$(\tau_{B1})(\tau_{B1}) = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-jB_x' d} & 0 & -e^{-jB_x' d} & 0 \\ 0 & e^{jB_x' d} & 0 & e^{jB_x' d} \\ e^{-jB_y' d} & 0 & e^{-jB_y' d} & 0 \\ 0 & -e^{jB_y' d} & 0 & e^{jB_y' d} \end{bmatrix} \quad (B-7)$$

And invoking Equation A-3, $(B_x' - B_y')d = \pm\pi/2$

$$(\tau_{B2})(\tau_{B1})(\tau_{B1}) = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-jB_x' d} e^{\mp j\theta} & 0 & -e^{-jB_x' d} e^{\mp j\theta} & 0 \\ 0 & e^{jB_x' d} e^{\pm j\theta} & 0 & e^{jB_x' d} e^{\mp j\theta} \\ \mp je^{-jB_x' d} e^{\mp j\theta} & 0 & e^{-jB_x' d} e^{\pm j\theta} & 0 \\ 0 & \pm je^{jB_x' d} e^{\pm j\theta} & 0 & \mp je^{jB_x' d} e^{\mp j\theta} \end{bmatrix} \quad (B-8)$$

These three matrices multiplied together produce, as they should, circularly polarized waves, but note that the first half of the phase shift $e^{\mp j\theta}$ is already present.

If the matrix of Equation B-8 is multiplied by $(\tau_{B2})^{-1}(\tau_{B2})$ there results

$$(\tau_{B2})^{-1}(\tau_{B2})(\tau_{B2})(\tau_{B1})(\tau_{B1}) = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-j3B_x' d} e^{\mp j2\theta} & 0 & -e^{-j3B_x' d} e^{\pm j2\theta} & 0 \\ 0 & e^{j3B_x' d} e^{\pm j2\theta} & 0 & e^{j3B_x' d} e^{\mp j2\theta} \\ \mp je^{-j3B_x' d} e^{\mp j2\theta} & 0 & \mp je^{-j3B_x' d} e^{\pm j2\theta} & 0 \\ 0 & \mp je^{j3B_x' d} e^{\pm j2\theta} & 0 & \mp je^{j3B_x' d} e^{\mp j2\theta} \end{bmatrix} \quad (B-9)$$

The phase shift is now the complete $\mp 2\theta$. The output of this matrix (for linear polarization input) is \pm circularly polarized. The function of the remaining two matrices (the $\lambda/4$ plate and transformation back to x,y coordinates) is to change these circular polarizations into linear polarization.

Equation B-4 can be put into scattering matrix notation by use of Equation 8, resulting in

$$\begin{aligned}
 S_{t11} &= S_{t13} = S_{t31} = S_{t33} = S_{t22} = S_{t24} = S_{t42} = S_{t44} = 0 \\
 S_{t12} &= t_{t44}/(t_{t22}t_{t44} - t_{t24}^2) \\
 S_{t14} &= -t_{t24}/(t_{t22}t_{t44} - t_{t24}^2) \\
 S_{t32} &= S_{t14} \\
 S_{t34} &= t_{t22}/(t_{t22}t_{t44} - t_{t24}^2) \\
 S_{t21} &= t_{t11} \\
 S_{t23} &= t_{t13} \\
 S_{t41} &= S_{t23} \\
 S_{t43} &= t_{t33}
 \end{aligned} \tag{B-10}$$

An examination of Figure B-1 convinces one that the device is reciprocal--the results are identical regardless of dissection or transmission. Thus $S_{tij} = S_{tji}$. If one uses Equation B-4 and carries out a somewhat messy multiplication, it is found that

$$t_{t22}t_{t44} - t_{t24}^2 = e^{j4(B_x' + B_y')d} \tag{B-11}$$

Substituting this result in Equation B-10 results in the scattering matrix

$$\begin{bmatrix} L_{E1x} \\ R_{E4x} \\ L_{E1y} \\ R_{E4y} \end{bmatrix} = \begin{bmatrix} 0 & t_{t11} & 0 & t_{t13} \\ t_{t11} & 0 & t_{t13} & 0 \\ 0 & t_{t13} & 0 & t_{t33} \\ t_{t13} & 0 & t_{t33} & 0 \end{bmatrix} \begin{bmatrix} R_{E1x} \\ L_{E4x} \\ R_{E1y} \\ L_{E4y} \end{bmatrix} \tag{B-12}$$

One of the remarkable things about this type of phase shifter is its broad band ability--in microwave usage such a phase shifter will maintain relative phase accuracy within ± 2 degrees over an entire waveguide band. At first glance this seems peculiar for $\lambda/4$ and $\lambda/2$ plates are all narrow band devices. Using Equation B-12 we can make a cursory examination of this behavior.

In a waveguide the input is $R_{E_{1x}}$ and the output $R_{E_{4x}}$ (as a choice). Since $R_{E_{1y}} = 0$, t_{t11} alone determines the phase shift. Suppose we choose an operating frequency 0.9 of the center frequency where the quarter- and half-wave plates are just that. Thus

$$(B_x' - B_y')d = \pm 0.9 \pi/2 \quad (B-13)$$

Then, from Equation B-4 (choosing Equation A-3 to be $+\pi/2$),

$$2t_{t11} e^{\frac{j4B_x'd}{\lambda}} = (1.80902 - j0.58779) \cos^2 \theta + 2 \sin^2 \theta (0.95106 - j0.30902) - 2 \sin \theta \cos \theta (0.61042 + j1.8787) \quad (B-14)$$

The phase set by a phase shifter is relative (i.e., one starts with an undetermined total phase shift and introduces so many degrees additional phase shift, reading from a dial calibrated in degrees rotation ($\times 2$) of the $\lambda/2$ plate). Thus, for $\theta = 0$, Equation B-14 comes out to be

$$2t_{t11} e^{\frac{j4B_x'd}{\lambda}} = 1.902 e^{-j18^\circ}$$

or

$$t_{t11} e^{\frac{j4B_x'd}{\lambda}} e^{j18^\circ} = 0.95 e^{j0^\circ} \quad (B-15)$$

With this information we can produce a table of 2θ , $|t_{t11}|$, and $4B_x d + 18^\circ + 4t_{t11}$ versus θ . It should be observed that at the center frequency, where the condition of Equation A-3 is met, that $|t_{t11}| = 1$, and the relative phase shift is just 2θ . In this case where $f = 0.9 f_o$, Equation B-12 shows some of $R_{E_{1x}}$ is siphoned off into $R_{E_{4y}}$. Since we have postulated just $\lambda/4$ followed by $\lambda/2$ followed by $\lambda/4$ plates, sans polarizers, this energy is lost for the purposes of the present discussion. In the case of a commercial phase shifter, the $\lambda/4$ plates are followed by a circular waveguide to rectangular transition which acts as a polarizer. Since the rectangular waveguide is below cutoff, E_{yy} is either absorbed or reflected--probably some of both, so Table B-1 should be a reasonable approximation to the real thing.

TABLE B-1. Phase Shift and Insertion Loss of Rotating Vane
 Phase Shifter at Center Frequency f_o and $0.9 f_o$
 Versus Angle of Rotation.

Angle of vane rotation, θ°	Phase shift, $^\circ$		Insertion loss, dB		Relative phase error, $^\circ$
	f_o	$0.9 f_o$	f_o	$0.9 f_o$	
0	0	0	0	0.45	0
2	4	4.15	0	0.45	0.15
4	8	8.31	0	0.45	0.31
6	12	12.46	0	0.45	0.46
8	16	16.60	0	0.45	0.60
10	20	20.74	0	0.45	0.74
12	24	24.87	0	0.35	0.87
14	28	28.99	0	0.35	0.99
16	32	33.10	0	0.35	1.10
18	36	37.28	0	0.35	1.28
20	40	41.29	0	0.35	1.29
22	44	45.37	0	0.35	1.37
24	48	49.44	0	0.26	1.44
26	52	53.49	0	0.26	1.49
28	56	59.53	0	0.26	1.53
30	60	61.56	0	0.26	1.56
32	64	65.57	0	0.26	1.57
34	68	69.58	0	0.18	1.58
36	72	73.57	0	0.18	1.57
38	76	77.55	0	0.18	1.55
40	80	81.53	0	0.18	1.53
42	84	85.49	0	0.09	1.49
44	88	89.44	0	0.09	1.44
46	92	93.39	0	0.09	1.39
48	96	97.33	0	0.09	1.33
50	100	101.27	0	0.09	1.27
52	104	105.21	0	0.09	1.21
54	108	109.14	0	0.09	1.14
56	112	113.06	0	0	1.06
58	116	116.99	0	0	0.99
60	120	120.92	0	0	0.92
62	124	124.84	0	0	0.88
64	128	128.77	0	0	0.77
66	132	132.70	0	0	0.70
68	136	136.63	0	0	0.63
70	140	140.56	0	0	0.56
72	144	144.49	0	0	0.49

From Table B-1 it is evident that the maximum phase error in a rotary vane phase shifter designed for a center frequency of 10 GHz would be about 1.6 degrees when operated at 9 GHz. Furthermore, the increase in insertion loss should be less than 0.5 dB. This substantiates the experimentally known fact that these devices are broad band.

To the extent that the quarter-wave and half-wave sections are not perfect, an actual device would show some deviation from Table B-1. However, if the departure from perfection is known (e.g., reflection and absorption coefficients), it should be straightforward to program a computer to produce the equivalent of Table B-1 for such a "real" device.

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 Albert Angelbeck (1)
 G. H. McLafferty (3)
- 3 United Aircraft Corporation, West Palm Beach, FL
 Dr. R. A. Schmidtke (1)
 Ed Pinsley (1)
- 4 University of Arizona, Optical Science Center, Tucson, AZ
 Dr. B. O. Seraphin (1)
 Dr. Francis Turner (1)
 Dr. William L. Wolfe (1)
 C. L. Blenman (1)
- 3 University of California, Lawrence Radiation Laboratory,
Livermore, CA
 Dr. Joe Fleck (1)
 Dr. R. E. Kidder (1)
 Dr. E. Teller (1)
- 1 University of Southern California, Los Angeles, CA (Seaver Science
Center, Dr. K. M. Lakin)
- 1 VARIAN Associates, San Carlos, CA (EIMAC Division, Jack Quinn)
- 1 Vought, Inc., Systems Division, Dallas, TX (F. G. Simpson)
- 3 Westinghouse Defense and Space Center, Baltimore, Md (R. A. Lee)
- 1 Westinghouse Electric Corporation, Research and Development
Laboratories, Pittsburgh, PA (E. P. Riedel)